MATH 141: Some Practice Final Problems

Key

Here are problems that cover the last two weeks of our class.

Remember the final is cumulative; you should look at Practice Midterm 1+2 and Midterm 1+2 as well.

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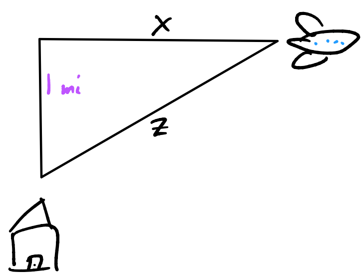
1. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.

① 

x: distance from plane to straight above the station

z: distance between plane and radar station

↑ functions of time!

② Given:

$$z = 2\,mi, \quad \frac{dx}{dt} = 500\,mi/h$$

Need:

$$\frac{dz}{dt}$$

③ $x^2 + 1^2 = z^2$ Pythagorean Thm

④ $$\frac{d}{dt}[x^2] + \frac{d}{dt}[1^2] = \frac{d}{dt}[z^2]$$

$$2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$$

missing from ②! Find what x is.

$$\frac{dz}{dt} = \frac{2x}{2z}\frac{dx}{dt} = \frac{\boxed{x}}{z}\cdot\frac{dx}{dt}$$

⑤ Now plug in z.

$$1^2 + x^2 = 2^2$$

$$x^2 = 4-1$$

$$x^2 = 3$$

$$\sqrt{x^2} = \pm\sqrt{3} \longrightarrow x = \sqrt{3}$$

⑥ $$\frac{dz}{dt} = \frac{x}{z}\cdot\frac{dx}{dt}$$

$$= \frac{\sqrt{3}\,mi}{\cancel{2}\,\cancel{mi}}\cdot\frac{\overset{250}{\cancel{500}}\,\cancel{mi}}{h}$$

$$= \boxed{250\ mi/h}$$

2. Short answer questions:

   (a) What are the two methods for finding local minimums and maximums called? State the weaknesses of one of the methods.

   ① First derivative test.

   ② Second derivative test; cannot determine local extrema in the case where $f'(c)$ DNE (the pointy corner extrema)

   (b) What is the method for finding absolute minimums and maximums called?

   Closed interval method.

   (c) What is the strength of **Part 2** of the Fundamental Theorem of Calculus?

   When evaluating integrals, instead of taking an infinite limit of a sum, you only need to take the antiderivative of the function you are integrating.

   (d) Find three particular antiderivatives of the function $f(x) = \sin(x)$.

$$\boxed{\begin{aligned} F_1(x) &= -\cos(x) + \pi \\ F_2(x) &= -\cos(x) + 3 \\ F_3(x) &= -\cos(x) - 1 \end{aligned}}$$

because

$$F_1'(x) = -(-\sin(x)) + 0 = \sin(x)$$

$$F_2'(x) = -(-\sin(x)) + 0 = \sin(x)$$

$$F_3'(x) = -(-\sin(x)) + 0 = \sin(x)$$

3. Suppose $f(x) = \dfrac{x}{x^2+1}$.

(a) Find all intervals on which $f(x)$ is increasing and decreasing.

① crit #'s

$$f'(x) = \frac{(x^2+1)\cdot\frac{d}{dx}[x] - x\,\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$

Quotient Rule

$$= \frac{x^2+1 - x\cdot 2x}{(x^2+1)^2}$$

$$= \frac{-x^2+1}{(x^2+1)^2}$$

ⓐ solve $f'(x) = 0$

$$\frac{-x^2+1}{(x^2+1)^2} = 0$$

$$-x^2+1 = 0$$

$$1 = x^2$$

$$\pm\sqrt{1} = x$$

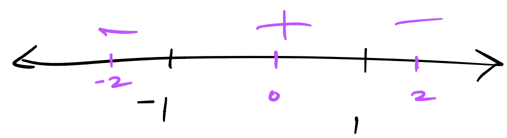
$$\boxed{x = \pm 1}$$

ⓑ $f'(x)$ DNE when $(x^2+1)^2 = 0$

but $x^2+1 > 0$ so not applicable.

(b) Find all local minimums and maximums.

② Sign diagram of f'

f' 

factor

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2} = \frac{-(x^2-1)}{(x^2+1)^2} = \frac{-(x-1)(x+1)}{(x^2+1)^2}$$

$$f'(-2) = \frac{-(-2-1)(-2+1)}{((-2)^2+1)^2} = \frac{-\cdot-\cdot-}{+} = -$$

$$f'(0) = \frac{-(0-1)(0+1)}{(0^2+1)^2} = \frac{-\cdot-\cdot+}{+} = +$$

$$f'(2) = \frac{-(2-1)(2+1)}{(2^2+1)^2} = \frac{-\cdot+\cdot+}{+} = -$$

Increasing on

$(-\infty, 2) \cup (2, \infty)$

Decreasing on

$(-2, 2)$

Local minimum of $f(-1) = \dfrac{-1}{(-1)^2+1} = -\dfrac{1}{2}$

Local maximum of $f(1) = \dfrac{1}{1^2+1} = \dfrac{1}{2}$

4. Evaluate the following expressions. If applicable, you are allowed to use Fundamental Theorem of Calculus.

(a) $\sum_{i=1}^{5} \frac{f(i)}{i}$ given that $f(x) = x^2$

$$\sum_{i=1}^{5} \frac{f(i)}{i} = \frac{f(1)}{1} + \frac{f(2)}{2} + \frac{f(3)}{3} + \frac{f(4)}{4} + \frac{f(5)}{5}$$

$$= 1 + \frac{2^2}{2} + \frac{3^2}{3} + \frac{4^2}{4} + \frac{5^2}{5}$$

$$= 1 + 2 + 3 + 4 + 5 = \boxed{15}$$

(b) $\int_1^5 x\,dx = \frac{1}{2}x^2 \Big|_1^5 = \frac{1}{2}\cdot 5^2 - \frac{1}{2}\cdot 1^2 = \frac{25}{2} - \frac{1}{2} = \frac{24}{2}$

$$= \boxed{12}$$

(c) $\lim_{x\to\infty} \frac{x+1}{\sqrt{x^4-2}} = \lim_{x\to\infty} \frac{\frac{x+1}{x^2}}{\frac{\sqrt{x^4-2}}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{\sqrt{\frac{x^4-2}{x^4}}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{\sqrt{1-\frac{2}{x^4}}}$

largest power is x^2

$$= \frac{\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2}}{\sqrt{\lim_{x\to\infty} 1 - 2\cdot\lim_{x\to\infty}\frac{1}{x^4}}} = \frac{0+0}{\sqrt{1-2\cdot 0}} = \boxed{0}$$

(d) $\lim_{x\to\infty}(x^5 - x^3) = \lim_{x\to\infty} x^2(x^3-1) = \infty\cdot\infty = \boxed{\infty}$

(e) $\int_{-1}^{1}(3x^2+4x+4)\,dx = 3\int_{-1}^{1} x^2\,dx + 4\int_{-1}^{1} x\,dx + \int_{-1}^{1} 4\,dx$

$$= \cancel{3}\cdot\frac{1}{\cancel{3}}x^3\Big|_{-1}^{1} + \overset{2}{\cancel{4}}\cdot\frac{1}{\cancel{2}}x^2\Big|_{-1}^{1} + 4x\Big|_{-1}^{1}$$

$$= \underbrace{1^3}_{F(b)} - \underbrace{(-1)^3}_{F(a)} + 2\cdot\left(1^2-(-1)^2\right) + 4\left(1-(-1)\right)$$

don't forget parentheses

$$= 1+1+2(1-1)+4(1+1) = 2+8 = \boxed{10}$$

(f) $\int_{\pi/2}^{\pi}\cos(x)\,dx$

$$\int_{\pi/2}^{\pi}\cos(x)\,dx = \sin(x)\Big|_{\pi/2}^{\pi} = \sin(\pi)-\sin\left(\frac{\pi}{2}\right) = 0-1=\boxed{-1}$$

(g) $\lim_{x\to\infty}\frac{x^2}{x^3+1}$

largest power

$$= \lim_{x\to\infty}\frac{\frac{x^2}{x^3}}{\frac{x^3+1}{x^3}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^3}} = \frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{1}{x^3}}$$

$$= \frac{0}{1+0}$$

$$= \boxed{0}$$

(h) $\int_{1}^{4}\frac{1}{\sqrt{x}}\,dx = \int_{1}^{4}x^{-\frac{1}{2}}\,dx = \frac{1}{\frac{1}{2}}x^{\frac{1}{2}}\Big|_{1}^{4}$

$$= 2\sqrt{x}\Big|_{1}^{4}$$

$$= 2\cdot\sqrt{4}-2\cdot\sqrt{1}$$

$$= 2\cdot 2-2 = \boxed{2}$$

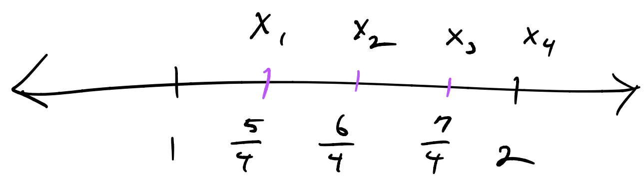
5. Suppose $f(x) = x^2$. Approximate the area underneath the curve on the interval $[1, 2]$ using four rectangles and right endpoints.

$a \quad b$

Only set up the sum; do not compute it.

$n = 4$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$



$$A = f(x_1)\cdot \Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= f\left(\tfrac{5}{4}\right)\cdot \tfrac{1}{4} + f\left(\tfrac{6}{4}\right)\cdot \tfrac{1}{4} + f\left(\tfrac{7}{4}\right)\cdot \tfrac{1}{4} + f(2)\cdot \tfrac{1}{4}$$

$$= \boxed{\frac{1}{4}\left(\left(\frac{5}{4}\right)^2 + \left(\frac{6}{4}\right)^2 + \left(\frac{7}{4}\right)^2 + 2^2\right)}$$

6. Consider the functions

$$g(x) = \int_0^x t^2 \, dt \qquad\qquad h(x) = \int_0^x \sin(t^3) \, dt$$

(a) What is the geometric meaning of the number $g(5)$?

The area between the curve and the x-axis of the function $f(x) = x^2$ on the interval $[0, 5]$

(b) What is the geometric meaning of the number $h(3)$?

The area between the curve and the x-axis of the function $f(x) = \sin(x^3)$ on the interval $[0, 5]$

(c) Find the derivative with respect to x of $g(x)$.

$$\frac{d}{dx} g(x) = \frac{d}{dx} \int_0^x t^2 \, dt = x^2$$

by FTC I

(d) Evaluate the following expression:

$$\frac{d}{dx}[g(x) + h(x)]$$

Differentiation formula (4) Section 2-3

$$\frac{d}{dx}[g(x) + h(x)] = \frac{d}{dx}[g(x)] + \frac{d}{dx}[h(x)]$$

$$= \frac{d}{dx} \int_0^x t^2 \, dt + \frac{d}{dx} \int_0^x \sin(t^3) \, dt$$

$$= \boxed{x^2 + \sin(x^3)}$$

7. Determine the intervals of concavity of

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$

① Find inflection points

$$f'(x) = 6x^2 - 18x + 12$$

$$f''(x) = 12x - 18 = 6(2x-3)$$

ⓐ Solve $f''(x) = 0$

$$6(2x-3) = 0$$

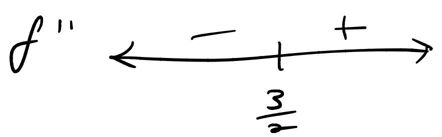
$$2x - 3 = 0$$

$$x = \frac{3}{2}$$

ⓑ find when $f''(x)$ DNE

not applicable, $f''(x)$ has domain $\mathbb{R}$.

② Sign diagram of f''



Concave up on $\left(\frac{3}{2}, \infty\right)$

Concave down on $\left(-\infty, \frac{3}{2}\right)$

$$f''(0) = 6 \cdot (2 \cdot 0 - 3) = -$$

$$f''(2) = 6 \cdot (2 \cdot 2 - 3) = +$$